The Solution of 2D Helmholtz Equations by Modified Explicit Group Iterative Method

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Abstract. The main aim of this paper is to examine a block iterative method known as the four point-Modified Explicit Group Modified Gauss Seidel (MEGGS) iterative method in solving 2D Helmholtz equations. The method is shown to be very much faster as compared to existing four-point block iterative method. In addition, by using an approximate equation based on the finite difference scheme, formulation and implementation of the proposed method to solve the problems are also presented. Numerical test and comparison with other existing four-point block iterative methods are given to illustrate the effectiveness of the proposed method.

Keywords: Helmholtz equations, Finite Difference (FD) scheme, Four point block scheme, Gauss Seidel (GS), Quarter-Sweep Iteration.

1 Introduction

The quarter-sweep iterative method was initiated recently by Othman and Abdullah [1] via the MEG iterative method to solve two-dimensional Poisson equations. Further studies to verify the effectiveness of the quarter-sweep iterative methods have been carried out; see Othman and Abdullah [2], Rakhimov and Othman [3] and Sulaiman et al. [4,5,6]. Eventually, the concept of this method is the extension of the half-sweep iterative method, which is inspired by Abdullah [7] through the Explicit Decoupled Group (EDG) iterative method to solve the same problem. Following to that, applications of the half-sweep iterative methods have also been discussed in Akhir et al. [8,9,10,11], Ibrahim and Abdullah [13], Muthuvalu and Sulaiman [14], Othman and Abdullah [15,16], Sulaiman et al. [17,18,19] and Yousif and Evans [20]. The basic idea of the half- and quarter-sweep iterative methods is to reduce the computational complexities during iteration process, since the implementation of the
half- and quarter-sweep iterations will only consider nearly half and quarter of all interior node points in a solution domain respectively.

Inspired by these findings, the purpose of this paper is to examine the application of the MEG iterative method with the GS method to form four point block iterative method called the four point-MEGGS method. To show the capability of the four point-MEGGS method, let us consider the following 2D Helmholtz equation with dirichlet boundary conditions on \( \Omega = [0, 1] \times [0, 1] \).

\[
U_{xx} + U_{yy} - \alpha U = f(x, y), \quad (x, y) \in \Omega = [0, 1] \times [0, 1]
\]

subject to the dirichlet boundary condition and to satisfy the exact solution \( U(x, y) = G(x, y) \), \((x, y) \in \Omega = \Omega \) where, \( f(x, y) \) is a given function with sufficient smoothness and \( \alpha \) is the non-negative constant. As mentioned in the previous paragraph, second order FD schemes will be considered to derive the full-, half-, and quarter-sweep five-point approximation equations for problem (1). Using second order central difference schemes, the full-, and quarter-sweep five-point approximation equations for problem (1) can be shown respectively as:

\[
U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1} - \left( 4 + h^2 \alpha \right) U_{i,j} = h^2 f_{i,j},
\]

and

\[
U_{i-2,j} + U_{i+2,j} + U_{i,j-2} + U_{i,j+2} - \left( 4 + 4h^2 \alpha \right) U_{i,j} = 4h^2 f_{i,j}
\]

Apart from Eqs. (2) and (3), another type of approximation, namely, the half-sweep FD case can be derived from the rotated FD approximation, which can be obtained by rotating the x-y axis clockwise 45° (Dahlquist and Bjorck [21]). Thus, the rotated FD approximation of Eq. (1) can be easily expressed as:

\[
U_{i+1,j+1} + U_{i-1,j-1} + U_{i+1,j-1} + U_{i-1,j+1} - \left( 4 + 2 \alpha h^2 \right) U_{i,j} = 2h^2 f_{i,j}
\]

Now it can be clearly seen that the application of either Eqs. (2), (3) or (4) to each interior node point will result in a large and sparse linear algebraic system

\[
Au = f
\]

where \( A \) and \( f \) are a square nonsingular matrix and a known vector, respectively. While \( u \) is an unknown vector to be determined. Actually, the solution of Eq. (5) can be obtained by direct or iterative methods. Since the coefficient matrix \( A \) of Eq. (5) is large and sparse, the iterative method is suitable to solve this type of problem and it can be solved either by point or block iterative methods. On the other hand, it was pointed out that block iterative methods converge faster than the point method, see Abdullah [7].

The outline of this paper is organized in the following way. In section two, of the paper will discuss the formulation and implementation for the combination between GS approach together with four point-MEG iterative method and some numerical results will be shown in third section to assert the performance of the proposed iterative method. Finally, section 4 contains some conclusion and directions of the future works.
2 Formulation and implementation of Four Point-MEGGS Iterative Method

As stated in the previous section, the four point-MEG iterative method has been introduced by Othman and Abdullah [1] in solving problem (1) by using a five point FD approximation equation in Eq.(3). In common, implementation of this method will be imposed onto solid node points, as shown in Fig. 1, until the convergence test criterion is satisfied. Afterwards, approximate values of the remaining node points at the FD networks, as shown in Fig. 1, will also be calculated directly by using the same steps in the FD scheme; see Abdullah [7], Akhir et al. [8,9,10,11], Ibrahim and Abdullah [13], Muthuvalu and Sulaiman [13], Othman and Abdullah [1,2,15,16], Sulaiman et al. [4,5,6,17,18,19], Rakhimov and Othman [3] and Yousif and Evans [20].

Based on Fig. 1, it can be seen that several completed groups of four points based on interior node points of type were presented. For the case of remaining same node points near the right and top boundaries, however, there exist several ungroup points scheme. According to previous studies (Othman and Abdullah [1,2] and Sulaiman et al. [6,18,19], Evans [23]), these ungroup points can be treated as a group of two points and the single point schemes as shown in Fig. 1.

Fig. 1 Four Point-MEGGS solution domain at m=16
For comparison purposes, other existing four point block iterative methods such as the four point-EDG and Explicit Group (EG) methods with GS approach namely EDGGS and EGGS are considered. Again, basically, these four point block iterative methods will be formulated by using the corresponding FD approximation equations in Eqs. (2), (3) and (4). From Fig. 1, and by using the approximation Eq. (3), consider any group of four points on the solution domain be used to construct the (4×4) linear system as follows

\[
\begin{bmatrix}
(4 + 4h^2\alpha) & -1 & 0 & -1 \\
-1 & (4 + 4h^2\alpha) & -1 & 0 \\
0 & -1 & (4 + 4h^2\alpha) & -1 \\
-1 & 0 & -1 & (4 + 4h^2\alpha)
\end{bmatrix}
\begin{bmatrix}
U_{i,j} \\
U_{i+2,j} \\
U_{i,j+2} \\
U_{i+2,j+2}
\end{bmatrix}
= \begin{bmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4
\end{bmatrix}
\tag{6}
\]

where

\[
S_1 = U_{i-2,j} + U_{i,j-1} - 4h^2 f_{i,j}, \\
S_2 = U_{i+4,j} + U_{i+2,j-2} - 4h^2 f_{i+2,j}, \\
S_3 = U_{i-2,j+2} + U_{i,j+4} - 4h^2 f_{i,j+2}, \\
S_4 = U_{i+4,j+2} + U_{i+2,j+4} - 4h^2 f_{i+2,j+2},
\]

Then, this system in Eq. (6) can be rewritten by multiplying the inverse of the coefficient matrix. As a result, the general scheme for the four point-MEGS method can be easily defined as

\[
\begin{bmatrix}
U_{i,j} \\
U_{i+2,j} \\
U_{i,j+2} \\
U_{i+2,j+2}
\end{bmatrix}
= \begin{bmatrix}
\beta & -1 & 0 & -1 \\
-1 & \beta & -1 & 0 \\
0 & -1 & \beta & -1 \\
-1 & 0 & -1 & \beta
\end{bmatrix}
^{-1}
\begin{bmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4
\end{bmatrix}
\tag{7}
\]

where \(\beta = 4 + 4h^2\alpha\).

The solution domain must be labeled for the four types of points (i.e., \(\bullet\), \(\bigcirc\), and \(\blacksquare\), as shown in Fig. 1. It can be clearly seen that evaluation of point type \(\bullet\) can be carried out independently. Also, approximately a quarter of the computational time and number of iterations of this method can be declined. Based on Eq. (9), we can define the four-point MEGGS iterative method by the following algorithm:

1. Divide the solution \(u\) domain into three types as in Fig. 1. Compute the values of \(4h^2\).
2. Iterate the intermediate solution \(U\) of point types \(\bullet\) using Eq. (9).
3. Check the convergence. If converge, evaluate the remaining of the points (i.e., □ and ○), respectively. Otherwise repeat the iteration cycle (i.e., go to step 2).
4. Display approximate solutions.

3 Numerical Experiments and Discussions

To examine the advantage of the four point-MEGGS method by using the quarter-sweep FD approximation in Eq. (3) based on second-order the FD scheme, three measurement parameters such as the number of iterations, execution time and maximum absolute error will be considered. As comparisons, the Full-Sweep Gauss-Seidel (FSGS) method acts as the comparison control of numerical results. Some numerical experiments were conducted in solving the following 2D Helmholtz equation as follows (Evans [23])

$$U_{xx} + U_{yy} - 10U = 6 - \left(20x^2 + 10y^2\right), \quad (x, y) \in \Omega = [0, 1] \times [0, 1] \quad (10)$$

The initial and boundary conditions and exact solution of the problem (10) are given by

$$U(x, y) = 2x^2 + y^2, \quad (x, y) \in \Omega = [0, 1] \times [0, 1] \quad (11)$$

All results of numerical experiments, obtained from implementation of the four point-MEGGS, EDGGS and MEGGS methods, have been summarized in Table 1. In the implementation mentioned above, the convergence criteria considered the tolerance error, \(\varepsilon = 10^{-10}\).

4 Conclusions

In this paper, we present the formulation of full-, half, and quarter-sweep approximation equations of problem (1) based on the second-order FD scheme as shown in Eqs. (2) till (4). Through numerical results collected Table 1, the findings clearly show that number of iterations have declined approximately 85.88 – 86.51%, 72.84 – 74.93% and 47.69 – 48.00% correspond to four point-MEGGS, EDGGS and EGGS methods compared to FSGS method. In fact, the execution time for all three four-point block methods are much faster approximately 34.48–50.29%, 82.76–87.34%, and 34.48–50.29% respectively than the FSGS method. In addition to these findings, the accuracy of the approximate solutions for the four point-MEGGS, EDGGS and EGGS methods are also in good agreement compared to the FSGS method. Overall, the numerical results show that the four point-MEGGS method is superior than other two four-point block methods. For future works, this study will be extended to investigate the usage of the Modified Successive Over Relaxation
approach (Young [22], Kincaid and Young [24], Akhir et al. [8,9,12,24,25], De Vogelaere [26]). Recently the development and implementation of the octo-sweep iterative method (Akhir et al. [27,28]) have been discussed to solve two-dimensional elliptic partial differential equations. Therefore, this concept of the octo-sweep iteration can be considered to solve problem (1) by using the corresponding octo-sweep FD approximation equation.

Table 1: Comparison of a number of iterations, execution time (seconds) and maximum absolute error for the iterative methods.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>Method</th>
<th>Numbers of iterations</th>
<th>Execution time (seconds)</th>
<th>Maximum absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FSGS</td>
<td>1326</td>
<td>0.29</td>
<td>6.8176e-10</td>
</tr>
<tr>
<td>32</td>
<td>4 EGGS</td>
<td>690</td>
<td>0.19</td>
<td>3.3179e-10</td>
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<tr>
<td>64</td>
<td>4 EDGGS</td>
<td>354</td>
<td>0.09</td>
<td>1.6075e-9</td>
</tr>
<tr>
<td></td>
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<td>187</td>
<td>0.05</td>
<td>8.1048e-10</td>
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<tr>
<td></td>
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<td>4910</td>
<td>1.38</td>
<td>2.7407e-10</td>
</tr>
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<td>0.73</td>
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<td>0.38</td>
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</tr>
<tr>
<td></td>
<td>FSGS</td>
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<td>15.43</td>
<td>1.1004e-10</td>
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<tr>
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<td>105.58</td>
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<tr>
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References