Different Growth Models in Population Dynamics and its Exponential Regression

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Abstract. Mathematical models of population growth have been constructed to provide an abstract of some significant aspect of true ecological situation. In this paper, we put some models where the parameters of the biological growth model systematically change over time. Verhust (1838) first proposed three-parameter model for the growth of single-species populations [1,5,9] that shows a logistic sigmoid growth curve for time. Pearl and reed (1920) independently published the same model, which includes mathematically an upper asymptote and a rate constant. The former and latter have been interpreted biologically as the carrying capacity of the environment and the intrinsic rate of natural increase in the population, respectively. Here we have discussed about some important growth models viz. Beverton-Holt Model, May model, Ricker Model, Birth-Death Model, Exponential Growth Model, Richard growth function, Gompertz growth function and Von Bertalanffy’s growth function.

Keywords: Population, Growth models, Beverton-Holt model, May model, Ricker model, Birth-Death model, Exponential growth model, Richard growth function.

1. Introduction
Population growth for a single-species population can be modeled in a variety of ways. Populations will grow exponentially if the per capita reproductive rate is constant and independent of population size. Simple difference equations used to model populations with discrete generations can generate complex behaviors including converging on an equilibrium population size, limit cycles, and chaos. The most common model used to estimate growth in populations with overlapping generations is the Verhulst–Pearl logistic equation where population growth stops at the carrying capacity. The increasing study of realistic and practically useful mathematical models in population biology [14,16,19], whether we are dealing with a human population with or without its age distribution, population of an endangered species, bacterial or viral growth and so on, is a reflection of their use in helping to understand the dynamic process involved and in making practical prediction.

2. Beverton-Holt Model
The Beverton-Holt model describes discrete-time populations:

\[ N_{t+1} = \frac{\lambda \times N_t}{1 + b \times N_t} \]

where \( \lambda \) represents the finite rate of increase and \( b \) is a positive constant. This models approaches a stable equilibrium point for all parameter values and the growth form is analogous to the continues-time logistic model

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3. May Model

Models which assume a nonlinear relationship between the growth rate and the population size can also generate realistic growth forms. One of the simplest nonlinear models (commonly referred to as the May model) is expressed as

\[ N_{t+1} = N_t \times e^{r\left(1 - \frac{N_t}{k}\right)} \]

where \( e \) is the base of the natural logarithm, \( r \) is the intrinsic rate of increase and \( k \) is the carrying capacity of the environment. This model assumes the relationship between \( r \) and \( N \) is nonlinear and the growth form exhibited by this model depends on the value of \( r \), \( k \) and the initial value of \( N \).

4. Ricker Model

The Ricker model is another nonlinear model which has a wide range of behaviors:

\[ N_{t+1} = \lambda \times N_t \times e^{b \times N_t} \]

This model, like equations of May model and Beverton-Holt model, can generate growth forms reaching a stable equilibrium point, can oscillate in a repeatable stable limit cycle, and can also generate chaotic, no repeating behavior, depending on the initial values of the parameters.

5. Birth-Death Model

The simplest model of population growth for species with continuous generations assumes that birth and immigration are combined and death and emigration are combined into a general addition and loss rate, respectively. These rates are constants and the model is represented as

\[ \frac{dN}{dt} = bN - dN \]

This implies \( N_t = N_0 \times e^{(b-d)\times t} \)
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where \( b \) is the instantaneous birth-immigration rate, \( d \) is the instantaneous mortality-emigration rate, \( N \) is the population size, and \( \frac{dN}{dt} \) represents the rate of change in \( N \). Thus if \( b > d \) the population grows exponentially while if \( b < d \) it dies out. This approach, due to Malthus in 1798, is fairly unrealistic.

6. Exponential Growth Model

Assume that in any time interval \((dt)\) an individual can be added to the population either through birth or immigration and in the same interval there is a probability of dying or emigrating, then the instantaneous rate of per capita growth will be

\[
R = b - d
\]

where \( r \) represents the intrinsic rate of growth for the population at time \( t \). The differential equation representing the increase in population size during successive time intervals is

\[
\frac{dN}{dt} = r \times N \quad \text{This implies} \quad N_t = N_0 \times e^{rt}
\]

where \( N \) represents the population size and \( r \) represents the populations intrinsic capacity for increase (also referred to as the Malthusian parameter). This model assumes that environmental resources are unlimited and \( r \) is a constant. The population will increase as long as \( r \) is positive (i.e. as long as \( b > d \)).

We will now examine the role of exponential growth functions in some real-world applications. In the following examples, assume that the population is modeled by an exponential growth function. Suppose that the population of a certain country grows at an annual rate of 2%. If the current population is 3 million, what will the population be in 10 years?

This is a future value problem. If we measure population in millions and time in years, then

\[
N_t = N_0 \times e^{r \times t}
\]
with $N_0 = 3$ and $r = 0.02$. Inserting these particular values into formula above we obtain

$$N_t = 3 \times e^{0.02t}$$

The population in 10 years is $N_{10} = 3 \times e^{(0.02)(10)} \approx 3.664208$ million.

In the same country as in above example how long will it take the population to reach 5 million? As before

$$N_t = 3 \times e^{0.02t}$$

Now we want to know when the future value $N_t$ of the population at some time $t$ will equal 5 million. Therefore, we need to solve the equation $N_t = 5$ for time $t$, which leads to the exponential equation

$$5 = 3 \times e^{0.02t}$$

This implies $t \approx 25.54128$.

Thus, it would take about 25.54 years for the population to reach 5 million.

7. Population Data Graph of India up to Year 2001

8. Richard Growth Function

Richard’s suggestion was to use the following equation which is also a special case of the Bernoulli differential equation

$$\frac{dN}{dt} = rN \left[ 1 - \left( \frac{N}{K} \right)^{\beta} \right]$$

which has the solution

$$N(t) = K \left[ 1 - e^{-\beta rt} \left( 1 - \left( \frac{N_0}{K} \right)^{-\beta} \right) \right]^{\frac{1}{\beta}}$$
9. Gompertz Growth Function

The Gompertz growth curve can be derived from the following form of the logistic equation as a limiting case:

\[
\frac{dN}{dt} = r N \left[ 1 - \left( \frac{N}{K} \right)^\beta \right]^\gamma = r' N \left( \frac{k^\beta N^\beta}{\beta} \right)^\gamma \quad \text{where} \quad r' = \frac{r}{K^\beta}
\]

Now,

\[
\lim_{\beta \to 0} \frac{K^\beta - N^\beta}{\beta} = \ln \left( \frac{K}{N} \right) \quad \text{and} \quad \lim_{\beta \to 0} (r') = r, \gamma > 0
\]

The growth rate model by the Gompertz function is given by

\[
\frac{dN}{dt} = r N \left[ \ln \left( \frac{N}{K} \right) \right]^\gamma
\]

Which has the solution

\[
N(t) = K \exp \left\{ \left[ \ln \left( \frac{N_0}{K} \right) \right]^{1-\gamma} + r' (-1)^\gamma (1 - \gamma) t \right\}^{\frac{1}{1-\gamma}}
\]

10. Von Bertalanffy’s Growth Function

Von Bertalanffy introduced his growth equation to model fish weight growth. Here the Verhulst logistic growth curve was modified to accommodate crude “metabolic types” based upon physiological reasoning. He proposed the form
given below which can be seen to be a special case of the Bernoulli differential equation:

\[
\frac{dN}{dt} = rN^\frac{2}{3} \left[ 1 - \left( \frac{N}{K} \right)^\frac{1}{3} \right]
\]

which has solution

\[
N(t) = K \left[ 1 + \left( 1 - \left( \frac{N_0}{K} \right)^\frac{1}{3} \right) e^{-\frac{1}{3}rKt} \right]^{3}
\]

11. Conclusion

This paper presents an overview of the mathematical models of population growth for single-species populations. This paper may be used to model population growth for species with non-overlapping generations. These models are represented mathematically by linear or nonlinear difference equations. In this paper total eight different models viz. Beverton-Holt Model, May model, Ricker Model, Birth-Death Model, Exponential Growth Model, Richard growth function, Gompertz growth function and Von Bertalanffy’s growth function has been discussed and the plot of number of species and exponentials of number of species also discussed. The population data of India has been collected and compared with exponential regression.

References

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