Analysis Blackman Window using Fractional Fourier Transform

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Abstract

The spectral parameters, SLA, HBW and SLFOR of Blackman window is analyzed, here the proposed method is Based on Fractional Fourier Transform by an Exponential approximation. This proposed derivations also holds good for generalization of FrFT with Fourier Transform (FT).

Keywords: Fractional Fourier transform, Blackman window.

1. Nomenclature

FT: Fourier Transform
FrFT: Fractional Fourier Transform
HBW: Half band width
MSLA: Maximum side lobe Attenuation
SLFOR: Side lobe fall-of-Ratio.

2. Introduction

In order to reduce the effects of spectral leakages in Harmonic analysis, windows are used [1]. window functions successfully used in the areas like interpolation factors to design Anti-Imaging filters, speech processing systems, digital filter design and beam forming [2]-[3]. windows are also useful to solve reconstructive errors which are objective functions to design the prototype filters [4]. windows are essentially Applicable in spectral analysis of signals[5]-[7]. In this proposed Derivatation of FrFT, An attempt is made to study the variations of window parameters like HBW, MSLA and MSLFOR by different values of fluid parameter of FrFT to FT at \( \alpha = \frac{\pi}{2} \). This paper is organized as follows. section-II gives an overview of FrFT, and mathematical model of Blackman window by using FrFT. Later conclusive remarks are discussed in section-III.

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3. Fractional Fourier Transform

Fractional Fourier Transform widely used in quantum mechanics and quantum optics [8]. spectral parameters of windows are tuned by using FrFT[9]. Fractional Fourier Analysis can obtain the mixed time and frequency components of signals[10]. it finds various applications like pattern recognition with some spatial distortion, Image representation, compression and noise processing [11]-[13]. FrFT used for Interpretation of sinusoidal signals and design of Digital FIR Filters[14]-[15].

The continuous –time Fractional Fourier Transform of a signal \( \omega(t) \) is defined through an interval [16]

\[
\omega_\alpha(u) = \int_{-\infty}^{\infty} \omega(t) K_\alpha(t, u) dt
\]

(1)

Where the transform kernel \( K_\alpha(t, u) \) of the FRFT is Given by

\[
K_\alpha(t, u) = \sqrt{\frac{1 - j \cot(\alpha)}{2\pi}} \exp \left( i \left( \frac{t^2 + u^2}{2} \right) \cot(\alpha) - j u \tan(\alpha) \right) \theta(t - u) \quad \text{if } \alpha \text{ is multiple of } \pi
\]

\[
= \delta(t + u) \quad \text{if } \alpha + \pi \text{ is a multiple of } 2\pi
\]

(2)

Where \( \alpha \) indicates rotation of angle of the Transformed signal for FrFT.
3.1 Blackman-window function

The Expression for Blackman window is [1]

\[
\omega(t) = 0.42 - 0.5 \cos(2\pi t) + 0.08 |t| < 1
\]

\[
\omega_w(u) = \int_{t_1}^{t_2} \omega(t) \sqrt{1 - j\cot(\alpha)} \exp \left( \frac{t^2}{2} \cot(\alpha) - i\text{cosec}(\alpha) \right) dt
\]

Let

\[
p = \frac{1 - j\cot(\alpha)}{2\pi} e^{\frac{u^2 \cot(\alpha)}{2}}
\]

Then equation-(4) becomes

\[
\omega_w(u) = p \int_{t_1}^{t_2} \omega(t) \exp \left( \frac{t^2}{2} \cot(\alpha) - i\text{cosec}(\alpha) \right) dt
\]

Substitute equation-(3) in equation-(6) then

\[
\omega_w(u) = p \int_{t_1}^{t_2} (0.42 - 0.5 \cos(2\pi t) + 0.08 \cos(4\pi t)) \exp \left( \frac{t^2}{2} \cot(\alpha) - i\text{cosec}(\alpha) \right) dt
\]

Equation-(7) divided into four parts like \(I_3, I_4, I_5, I_6\) where

\[
I_3 = \int_{t_1}^{t_2} (0.42) \exp \left( \frac{t^2}{2} \cot(\alpha) - i\text{cosec}(\alpha) \right) dt - (8)
\]

\[
I_4 = \int_{t_1}^{t_2} (0.5) \cos(2\pi t) \exp \left( \frac{t^2}{2} \cot(\alpha) - i\text{cosec}(\alpha) \right) dt
\]

\[
I_5 = \int_{t_1}^{t_2} (0.5) \cos(4\pi t) \exp \left( \frac{t^2}{2} \cot(\alpha) - i\text{cosec}(\alpha) \right) dt
\]

\[
I_6 = \int_{t_1}^{t_2} (0.08) \cos(4\pi t) \exp \left( \frac{t^2}{2} \cot(\alpha) - i\text{cosec}(\alpha) \right) dt
\]

Now solving for \(I_3\)

\[
I_3 = p \left( \int_{t_1}^{t_2} 0.42 \exp \left( \frac{t^2}{2} \cot(\alpha) - i\text{cosec}(\alpha) \right) dt \right) - (11)
\]

Now Multiply both sides with \(e^{-\frac{t^2}{2}\cot(\alpha)}\), you will get

\[
e^{-\frac{t^2}{2}\cot(\alpha)}I_3 = p * 0.42 \left( \int_{t_1}^{t_2} \exp \left( \frac{t^2}{2} \cot(\alpha) - i\text{cosec}(\alpha) \right) e^{-\frac{t^2}{2}\cot(\alpha)} dt \right) - (12)
\]

\[
e^{-\frac{t^2}{2}\cot(\alpha)}I_3 = p * 0.42 \left( \int_{t_1}^{t_2} \exp \left( -i\text{cosec}(\alpha) \right) dt \right) - (13)
\]

Now integrating and applying limits on both sides

\[
e^{-\frac{t^2}{2}\cot(\alpha)}I_3 = p * 0.42 \left( \frac{e^{-i\text{cosec}(\alpha)}}{-\text{cosec}(\alpha)} - \frac{e^{-i\text{cosec}(\alpha)}}{-\text{cosec}(\alpha)} \right)
\]

\[
e^{-\frac{t^2}{2}\cot(\alpha)}I_3 = p * 0.42 \left( \frac{e^{-i\text{cosec}(\alpha)}}{-\text{cosec}(\alpha)} - \frac{e^{-i\text{cosec}(\alpha)}}{-\text{cosec}(\alpha)} \right)
\]

Now apply limits \(\int_{t_3}^{t_2} e^{-\frac{t^2}{2}\cot(\alpha)} I_3 dt\) on both sides, results to

\[
\int_{t_1}^{t_2} e^{-\frac{t^2}{2}\cot(\alpha)} I_3 dt = \int_{t_1}^{t_2} p \left( \frac{e^{-i\text{cosec}(\alpha)}}{-\text{cosec}(\alpha)} - \frac{e^{-i\text{cosec}(\alpha)}}{-\text{cosec}(\alpha)} \right) dt
\]

\[
\int_{t_1}^{t_2} e^{-\frac{t^2}{2}\cot(\alpha)} I_3 dt = p \left( \frac{e^{-i\text{cosec}(\alpha)}}{-\text{cosec}(\alpha)} - \frac{e^{-i\text{cosec}(\alpha)}}{-\text{cosec}(\alpha)} \right) t_2 - t_1
\]

Let \(I_2 = \int_{t_1}^{t_2} e^{-\frac{t^2}{2}\cot(\alpha)} dt\)

According to [17]

\[
\exp \left( -i \frac{t^2}{2} \cot(\alpha) \right) = (1 - i \frac{t^2}{2} \cot(\alpha)) \quad (19)
\]

Substitute equation-(19) in equation-(18) and integrating we get

\[
I_2 = \int_{t_1}^{t_2} ((1 + i \frac{t^2}{2} \cot(\alpha)) dt
\]

\[
I_2 = \int_{t_1}^{t_2} 1 dt + \frac{i \cot(\alpha)}{2} \int_{t_1}^{t_2} t^2 dt
\]

\[
I_2 = (t_2 - t_1) + \frac{i \cot(\alpha)}{6} (t_2^3 - t_1^3)
\]

Finally

\[
I_2 = \frac{0.42 \left( \frac{e^{-i\text{cosec}(\alpha)}}{-\text{cosec}(\alpha)} - \frac{e^{-i\text{cosec}(\alpha)}}{-\text{cosec}(\alpha)} \right)}{(t_2 - t_1) + \frac{i \cot(\alpha)}{6} (t_2^3 - t_1^3)}
\]

Now solving for \(I_4\)

\[
I_4 = \int_{t_1}^{t_2} (0.5) \cos(2\pi t) \exp \left( \frac{t^2}{2} \cot(\alpha) - i\text{cosec}(\alpha) \right) dt
\]

But

\[
\cos(2\pi t) = \exp(2\pi t) + \exp(-2\pi t)
\]

Substitute equation-(25) in equation-(24) then

\[
I_4 = \int_{t_1}^{t_2} ((0.5) \left( \frac{\exp(2\pi t) + \exp(-2\pi t)}{2} \right)) \exp \left( \frac{t^2}{2} \cot(\alpha) - i\text{cosec}(\alpha) \right) dt
\]
\[ \int_0^{2\pi} \frac{1}{\sin^2 \theta} \, d\theta = \pi \]
\[ x_1 = \int_{t_1}^{t_2} \left( 0.04 \left( \frac{\exp(i4\pi t_2)}{i2\pi} - \frac{\exp(i4\pi t_1)}{i2\pi} \right) \right) dt \]

And
\[ x_2 = \int_{t_1}^{t_2} \left( e^{-\frac{t^2}{2}\cot(a)} \right) dt \]

Now solving for \( x_1 \)
Integrating \( x_1 \) and applying limits
\[ x_1 = 0.04 \left( \frac{\exp(i4\pi t_2)}{i2\pi} - \frac{\exp(i4\pi t_1)}{i2\pi} \right) (t_2 - t_1) - (49) \]

Now solving for \( x_2 \)
\[ x_2 = \int_{t_1}^{t_2} \left( e^{-\frac{t^2}{2}\cot(a)} \right) dt - - - - - - - -(50) \]

According to [17]
\[ e^{-\frac{t^2}{2}\cot(a)} = 1 - i \frac{t^2}{2} \cot(a) - - - - - - - - (51) \]

Substitute equation-(51) in equation-(50) to get
\[ x_2 = \int_{t_1}^{t_2} \left( 1 - i \frac{t^2}{2} \cot(a) \right) dt - - - - - - - -(52) \]

Integrating and applying limits to get
\[ x_2 = \int_{t_1}^{t_2} dt - \frac{i \cot(a)}{2} \int_{t_1}^{t_2} t^2 dt - - - - - - - (53) \]
\[ x_2 = (t_2 - t_1) - \frac{i \cot(a)}{6} (t_2^3 - t_1^3) - - - - - - - (54) \]

Finally \( I_5 \)
\[ I_5 = \frac{0.04 \left( \frac{\exp(i4\pi t_2)}{i2\pi} - \frac{\exp(i4\pi t_1)}{i2\pi} \right) (t_2 - t_1)}{(t_2 - t_1) - \frac{i \cot(a)}{6} (t_2^3 - t_1^3)} - (55) \]

Finally
\[ \omega_n(u) = I_3 - I_4 + I_5 - - - - - - - - - (56) \]

Thus equation-(56) is the FRFT based Term Blackman window.

When substitute \( a = \frac{\pi}{2} \) where \( a = 1 \) in equation-(56) results to generalized Fourier Transform based Blackman window. and its spectral responses are observed for different values of ‘a’, which are shown from Fig:1 to Fig:8.then the spectral values are Tabulated.
Table:
Spectral Parameters of FrFT Based Blackman window for variations in ‘a’.

<table>
<thead>
<tr>
<th>a</th>
<th>HBW in dB</th>
<th>MSLA in dB</th>
<th>SLFOR in dB</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.935</td>
<td>0.0185</td>
<td>-58.5</td>
<td>-74.09</td>
</tr>
<tr>
<td>0.94</td>
<td>0.0185</td>
<td>-60.5</td>
<td>-71.50</td>
</tr>
<tr>
<td>0.95</td>
<td>0.0185</td>
<td>-64.0</td>
<td>-71.38</td>
</tr>
<tr>
<td>0.96</td>
<td>0.0185</td>
<td>-61.8</td>
<td>-70.71</td>
</tr>
<tr>
<td>0.97</td>
<td>0.0185</td>
<td>-60.2</td>
<td>-70.15</td>
</tr>
<tr>
<td>0.98</td>
<td>0.0185</td>
<td>-59.0</td>
<td>-69.72</td>
</tr>
<tr>
<td>0.99</td>
<td>0.0185</td>
<td>-58.3</td>
<td>-68.58</td>
</tr>
<tr>
<td>1</td>
<td>0.0185</td>
<td>-58.1</td>
<td>-69.48</td>
</tr>
</tbody>
</table>

4. Conclusion

From the study of Exponential derivation of FrFT Blackman window, it is possible to obtain better spectral window parameters like HBW, MSLA and SLFOR for Blackman window. The MSLA varies from -58.1 dB to -64.0 dB by different values of ‘a’. From the table it is observed that MSLA and SLFOR increases between ‘a’ value of 0.935 and 0.95. From there both MSLA and SLFOR decreases between ‘a’ value of 0.96 to 1. And it is also observed that HBW remains unchanged for all values of ‘a’. one of the property of FrFT is Generalization of FrFT to FT i.e. when $\alpha = \frac{a}{2}$ where $a=1$; then The FrFT should equals to FT

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References


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