A Best Proximity Theorem for Some General Contractive Pair of Maps

Bhagwati Prasad
Department of Mathematics
Jaypee Institute of Information Technology
A-10, Sector-62, Noida-201307, India
b_prasad10@yahoo.com; bhagwati.prasad@jiit.ac.in

Abstract—The intent of the paper is to study semi-cyclic type contraction condition for a pair of maps \( (S, T) \). Our aim is to establish an existence theorem for common fixed points and best proximity points for such a pair in Banach spaces. The results obtained herein extend some recent results.

Index Terms—Fixed point; best proximity point; cyclic contraction.

I. INTRODUCTION

Let \( T \) be a self map of a nonempty set \( X \). A point \( x \in X \) such that \( Tx = x \) is called a fixed point of the map \( T \). The fixed points are of vital importance since most of the equations arising in the modeling of various physical formulations can easily be transformed into a fixed point equation and a fixed point or an approximate fixed point of the map \( T \) provides the solution or approximate solution to such problems. If \( T \) is a non-self-mapping, it may be possible that the fixed point equation \( Tx = x \) has no solution. In such a case the best proximity point theorems analyze the existence of an optimal approximate solution. Kirk et al [6] introduced 2-cyclic contraction and obtained the best proximity points of the map. Eldred and Veeramani [3] defined a more generalized notion of cyclical maps and established a unique best proximity point for the mapping \( T \) in a uniformly convex Banach space. Subsequently, a number of extensions and generalizations of their results appeared in [1]-[2], [4]-[5] and [7] and several references thereof. Recently, Gabeleh and Abkar [4] introduced the notion of a semi-cyclic contraction pair by taking two self maps. Our aim is to establish an existence and convergence result for the best proximity points for a semi-cyclic contraction pair \( (S, T) \) in Banach spaces.

II. PRELIMINARIES


Let \( (X, d) \) be a complete metric space and \( T : A \cup B \rightarrow A \cup B \), where \( A \) and \( B \) are any two nonempty closed subsets of \( X \). The map \( T \) is said to be cyclic if \( T(A) \subseteq B \) and \( T(B) \subseteq A \). Assume further that there exists \( 0 \leq \alpha < 1 \) such that

\[
d(Tx, Ty) \leq \alpha d(x, y), \quad x \in A, \ y \in B.
\]
It follows that \( A \cap B \neq \emptyset \) and that the cyclic map \( T \) has a unique fixed point in \( A \cap B \) (see [6]).

According to [3], a self mapping \( T \) on \( A \cup B \) is said to be a cyclic contraction if \( T \) is cyclic and satisfies the following condition for all \( x \in A, \ y \in B \) and \( 0 \leq \alpha < 1 \).

\[
d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha) d(A, B) \quad (2)
\]

It is to be noticed that in this case the fixed point of \( T \) does not exist (since \( A \cap B = \emptyset \)), however, one may discover a possibility of a best proximity point, that is, an \( x \) in \( A \cup B \) such that

\[
d(x, Tx) = d(A, B) = \inf \{ d(x, y) : x \in A, \ y \in B \}.
\]

In [4], the notion of \((S, T)\) semi-cyclic contraction pair is introduced by taking two self maps \( S \) and \( T \) on \( A \cup B \) such that \( S(A) \subseteq B, T(B) \subseteq A \) and \( S, T \) satisfy the following condition for all \( x \in A, \ y \in B \) and \( 0 \leq \alpha < 1 \).

\[
d(Sx, Ty) \leq \alpha d(x, y) + (1 - \alpha) d(A, B) \quad (3)
\]

It is remarkable that a semi-cyclic contraction pair reduces to a cyclic contraction when we put \( S = T \).

III. MAIN RESULTS

A. Proposition 3.1

Let \( A \) and \( B \) be nonempty subsets of a metric space \( X \) and \( S, T : A \cup B \to A \cup B \) satisfy \( S(A) \subseteq B, T(B) \subseteq A \) and

\[
d(Sx, Ty) \leq \alpha d(x, y) + \beta d(Sx, Ty) + \gamma d(A, B) \quad (4)
\]

for all \( x, y \in A \cup B \), where \( \alpha, \beta, \gamma \geq 0 \) and \( \alpha + 2\beta + \gamma < 1 \).

Consider \( x_0 \in A \), and define: \( x_{n+1} = Ty_n, \ y_n = Sx_n, \ n = 0, 1, 2, ... \). Then \{\( x_n \)\} and \{\( y_n \)\} are sequences in \( A, B \), respectively. Moreover, \( d(x_n, Sx_n) \to d(A, B) \) and \( d(y_n, Ty_n) \to d(A, B) \).

Proof. First, we note that

\[
d(x_n, Sx_n) = d(Ty_{n-1}, Sx_n)
\]

\[
\leq \alpha d(x_n, y_{n-1}) + \beta [d(x_n, Sx_n) + d(y_{n-1}, Ty_{n-1})] + \gamma d(A, B)
\]

Therefore,

\[
d(Ty_{n-1}, Sx_n) \leq \frac{\alpha}{1 - \beta} d(x_n, y_{n-1}) + \frac{\beta}{1 - \beta} d(y_{n-1}, Ty_{n-1}) + \frac{\gamma}{1 - \beta} d(A, B)
\]

\[
\leq \frac{\alpha + \beta}{1 - \beta} d(Sx_{n-1}, Ty_{n-1}) + \frac{\gamma}{1 - \beta} d(A, B)
\]

Now, if \( s = (\alpha + \beta)/(1 - \beta) \), then

\[
d(Ty_{n-1}, Sx_n) \leq sd(Sx_{n-1}, Ty_{n-1}) + (1 - s)d(A, B)
\]

\[
\leq s^2 d(Sx_{n-1}, Ty_{n-1}) + (1 - s^2)d(A, B)
\]

\[
\leq ... \leq s^{2n} d(x_1, y_0) + (1 - s^{2n})d(A, B)
\]

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Hence \( d(x_n, Sx_n) \to d(A, B) \).
Similarly, it can be shown that \( d(y_n, Ty_n) \to d(A, B) \).

**B. Proposition 3.2**

Let \((S, T)\) be a semi-cyclic pair of maps satisfying (4) and sequences \(\{x_n\}\) and \(\{y_n\}\) are generated as follows:

\[
x_{n+1} = Ty_n \quad \text{and} \quad y_n = Sx_n, \quad n = 0, 1, 2, \ldots
\]

If both \(\{x_n\}\) and \(\{y_n\}\) have a convergent subsequence in \(A\) and \(B\), respectively, then there exist \(x \in A\) and \(y \in B\) such that

\[
d(x, Sx) = d(A, B) = d(y, Ty).
\]

**Proof.** Let \(\{y_{n_k}\}\) be a subsequence of \(\{y_n\}\) such that \(y_{n_k} \to y\).

Since \(d(A, B) \leq d(Ty_{n_k}, y) \leq d(y, y_{n_k}) + d(y_{n_k}, Ty_{n_k})\),
from Proposition 3.1, we obtain \(d(y, Ty_{n_k}) \to d(A, B)\).

Also, we have

\[
d(A, B) \leq d(Ty_{n_k}, y_{n_k}) = d(Ty, Sx_{n_k})
\]

\[
\leq \alpha d(x_{n_k}, y) + \beta d(x_{n_k}, Sx_{n_k}) + d(y, Ty) + \gamma d(A, B)
\]

\[
\leq \alpha d(Ty_{n_k}, y) + \beta d(Ty_{n_k}, y_{n_k}) + d(y, Ty) + \gamma d(A, B).
\]

On letting \(k \to \infty\), we obtain \(d(Ty, y) = d(A, B)\).

On the similar steps, we can prove \(d(x, Sx) = d(A, B)\).

We need following results of [3] in the sequel.

**C. Lemma 3.1 [3]**

Let \(X\) be a uniformly convex Banach space, \(A\) a nonempty closed convex subset and \(B\) a nonempty closed subset of it. Let \(\{x_n\}\), \(\{z_n\}\) be two sequences in \(A\), and \(\{y_n\}\) be a sequence in \(B\) such that

(i) \(\|z_n - y_n\| \to \|A - B\|\)

(ii) \(\forall \varepsilon > 0, \exists N_0\) such that for all \(m > n \geq N_0\):

\[
\|x_m - y_n\| \leq \|A - B\|
\]

Then for every \(\varepsilon > 0\) there exists \(N_1\) such that for all \(m > n \geq N_1\) we have \(\|x_m - z_n\| \leq \varepsilon\).

**D. Lemma 3.2 [3]**

Let \(X\) be a uniformly convex Banach space, \(A\) a nonempty closed convex subset and \(B\) a nonempty closed subset of it. Let \(\{x_n\}\), \(\{z_n\}\) be two sequences in \(A\), and \(\{y_n\}\) be a sequence in \(B\) such that

(i) \(\|x_n - y_n\| \to \|A - B\|\).

(ii) \(\|z_n - y_n\| \to \|A - B\|\).

Then \(\|x_n - z_n\| \to 0\).
Now we present our main theorem.

Theorem 3.1. Let $A, B$ be two nonempty closed convex subsets of a uniformly convex Banach space $X$. Let $(S, T)$ be a semi-cyclic mapping satisfying (4).

(i) If $\|A - B\| = 0$, then $S, T$ have a unique common fixed point in $A \cap B$.

(ii) If $\|A - B\| > 0$, then each mapping has a unique best proximity point.

Additionally, we can approximate fixed point or the best proximity point through some iterative sequences.

Proof. We first assume $\|A - B\| = 0$. Then for all $x \in A, y \in B$

$$\|Tx - Ty\| \leq \alpha \|d(x, y)\| + \beta \|x - Sx\| + \|y - Ty\|$$

The sequence $\{z_n\}_{n \geq 1}$ in $A \cup B$ is defined as follows:

$$z_n = \begin{cases} Ty_k & n = 2k \\ Sx_k & n = 2k - 1 \end{cases}$$

To prove that $\{z_n\}_{n \geq 1}$ is a Cauchy sequence in $A \cup B$.

If $n = 2k$, we have

$$\|z_{n+1} - z_n\| = \|Sx_{k+1} - Ty_k\|$$

$$\leq \alpha \|d(x_{k+1}, y_k)\| + \beta \|x_{k+1} - Sx_k\| + \|y_k - Ty_k\|$$

$$\leq \alpha \|d(Ty_k - Sx_k) + \beta \|Ty_k - Sx_k\| + \|Sx_k - Ty_k\|$$

$$\leq \frac{\alpha + \beta}{1 - \beta} \|Ty_k - Sx_k\|.$$

Let $s = \frac{\alpha + \beta}{1 - \beta}$, then

$$\|z_{n+1} - z_n\| = \|Sx_{k+1} - Ty_k\| \leq s \|Ty_k - Sx_k\|$$

$$\leq s^k \|y_k - x_k\| \leq \ldots \leq s^k \|y_1 - x_1\| \to 0 \quad \text{as} \quad k \to \infty.$$  

Proceeding similarly for $n = 2k - 1$, we can draw the same conclusion, so that for $m > n$,

$$\|z_m - z_n\| \leq \sum_{k=n}^{m-1} s^k \|y_1 - x_1\| \to 0, \quad n, m \to \infty$$

Then there exists $z \in A \cup B$ such that $z_n \to z$. Assume that $z \in A$. Since $\{z_{2k-1}\} \subseteq B$, it follows that $z \in B$, and finally $z \in A \cap B$. In case that $z \in B$, the same argument again shows that $z \in A \cap B$.

On the other hand,

$$\|z - Tz\| = \lim_k \|y_k - Tz\| = \lim_k \|Sx_k - Tz\|$$

$$\leq \alpha \|x_k - z\| + \beta \|x_k - Sx_k\| + \|z - Tz\|,$$

and thus

$$\|z - Tz\| \leq \beta \|z - Tz\|,$$ a contradiction.

This implies that $Tz = z$.

Similarly, we see that $Sz = z$.
Hence $T, S$ have a common fixed point.

To prove uniqueness, we take another common fixed point $z$ of the maps (if exists).
In fact, if $Tw = w = Sw$ for some $w \in A \cap B$, then
\[
\|z - w\| = \|Tz - Sw\| \leq \alpha \|z - w\| + \beta \|z - Sz\| + \|w - Tw\|
\]
Thus $z = w$. This completes the proof of part (i).

To prove (ii), assume $\|A - B\| > 0$.
Since $(S, T)$ is a semi-cyclic contraction pair satisfying (4), we have
\[
\|y_n - Ty_n\| = \|Sx_n - Ty_n\|
\leq \alpha \|x_n - y_n\| + \beta \|x_n - Sx_n\| + \gamma \|y_n - Ty_n\| + \|A - B\|
\]
Using Proposition 3.1 in it, we get
\[
\|y_n - x_{n+1}\| \rightarrow \|A - B\|
\]
On the similar lines, we obtain $\|y_{n+1} - x_{n+1}\| \rightarrow \|A - B\|
Using Lemma 3.2, we have $\|y_n - y_{n+1}\| \rightarrow 0$.
Similarly, $\|x_n - x_{n+1}\| \rightarrow 0$.

Now we claim that for every $\varepsilon > 0$ there exists $N_0$ such that for all $m > n > N_0$, we have
\[
\|y_m - Ty_n\| = \|y_m - x_{n+1}\| \leq \|A - B\| + \varepsilon.
\]
We suppose the contrary, then there exists $\varepsilon > 0$ such that for all $k \geq 1$ there exist $m_k > n_k \geq k$ for which
\[
\|y_{m_k} - Ty_{n_k}\| \geq \|A - B\| + \varepsilon.
\]
This $m_k$ may be chosen in such a way that it is the least integer greater than $n_k$ to satisfy the above inequality.
Now
\[
\|A - B\| + \varepsilon \leq \|y_{m_k} - Ty_{n_k}\| \leq \|y_{m_k} - y_{m_{k+1}}\| + \|y_{m_{k+1}} - Ty_{n_k}\|
\leq \|y_{m_k} - y_{m_{k+1}}\| + \|A - B\| + \varepsilon
\]
Hence $\|y_{m_k} - Ty_{n_k}\| \rightarrow \|A - B\| + \varepsilon$. Then
\[
\|y_{m_k} - Ty_{n_k}\| \leq \|y_{m_k} - y_{m_{k+1}}\| + \|y_{m_{k+1}} - Ty_{n_k}\| \leq \|y_{m_k} - y_{m_{k+1}}\| + \varepsilon
\]
\[
\leq \|y_{m_k} - y_{m_{k+1}}\| + \varepsilon \leq \|y_{m_k} - y_{m_{k+1}}\| + \varepsilon \leq \|A - B\| + \varepsilon,
\]
letting $k \rightarrow \infty$ in above, we obtain
\[
\|A - B\| + \varepsilon \leq \|A - B\| + \varepsilon
\]
which is a contradiction.
Hence, $\{y_n\}$ is a Cauchy sequence by Lemma 3.1.
So, there exists $y \in B$ such that $\{y_n\} \rightarrow y$.
From Proposition 3.2, $\|y - Ty\| = \|A - B\|$. Similarly,
we can be prove \( \{x_n\} \rightarrow x \in A \) and \( \|x - Sx\| = \|A - B\| \).

For the uniqueness, let \( w \in A \) be such that \( \|w - Sw\| = \|A - B\| \). Since
\[
\|A - B\| \leq \|T^*Sx - Sx\|
\leq \alpha \|Sx - x\| + \beta \|Sx - Sx\| + \|x - Sx\| + \gamma \|A - B\|
\leq (\alpha + 2\beta + \gamma) \|A - B\| \leq \|A - B\|
\]
It follows that \( T^*Sx - Sx = \|x - Sx\| \). This in turn establishes \( TSx = x \). Similarly, we see that \( TSw = w \).

Now if \( w \neq x \), then \( \|x - Sw\| > \|A - B\| \), from which we obtain
\[
\|Sx - w\| = \|Sx - TSx\| \leq \alpha \|x - Sw\| + \beta \|x - Sx\| + \|Sw - TSx\| + \gamma \|A - B\|
\leq \alpha \|x - Sw\| + (\beta + \gamma) \|A - B\|
\leq \alpha \|x - Sw\| + (1 - \alpha \|A - B\|)
\]
Again \( \|x - Sw\| = \|TSx - Sw\| \)
\[
\leq \alpha \|Sx - w\| + \beta \|Sx - TSx\| + \|w - Sw\| + \gamma \|A - B\|
\leq \alpha \|Sx - w\| + (2\beta + \gamma) \|A - B\|
\leq \alpha \|Sx - w\| + (1 - \alpha) \|Sx - w\|
\]
Thus we have, \( \|Sx - w\| \leq \|Sx - w\| \), a contradiction.
Hence the proof is completed.

\[E. \text{ Corollary 3.1} \[4\]\]
Let \( A, B \) be two nonempty closed convex subsets of a uniformly convex Banach space \( X \). Let \((S, T)\) be a semi-cyclic contraction mapping.
(i) If \( \|A - B\| = 0 \), then \( S, T \) have a unique common fixed point in \( A \cap B \).
(ii) If \( \|A - B\| > 0 \), then each mapping has a unique best proximity point.
Moreover either of fixed point or best proximity points can be approximated by some iterative sequences.

Proof. It follows from Theorem 3.1 when \( \beta = 0 \) and \( \gamma = 1 - \alpha \).

\[C O N C L U S I O N\]
We obtain an existence and convergence result for the common fixed points and best proximity points for a semi-cyclic contraction pair in Banach spaces. Such results analyze the existence of an optimal approximate solution in case a fixed point equation has no solution. The results obtained herein extend the result of Gabeleh and Abkar [4].

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REFERENCES


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